Written Exam for the M.Sc. in Economics summer 2011

Asset Pricing Theory

Final Exam

3 June 2011

(3-hour closed book exam)

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by “eksamen på dansk” in brackets, you must write your exam paper in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students’ self-service system.

The Exam consists of three problems each weighted by $33^1/_3$%

Problem 1 — Pricing of derivative security

Suppose that a stock price S_t follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$

Where z_t is a standard Brownian motion (Wiener process). The stochastic differential equation describing the dynamics of $G(S_t, t)$, a continuous and twice differentiable function of the stock price and time, is given by Itô's lemma:

$$dG(S_t, t) = \left(\frac{\partial G(S_t, t)}{\partial t} + \frac{\partial G(S_t, t)}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 G(S_t, t)}{\partial S_t^2}\sigma^2 S_t^2\right)dt + \frac{\partial G(S_t, t)}{\partial S_t}\sigma S_t dz_t$$

(a) Show that the process followed by the variable $G(S_t, t) = (S_t)^3$ is also a geometric Brownian motion.

Consider a security which pays off an amount equal to $(S_T)^3$ at time T, where S_T denotes the value of the stock at time T. Let $H(S_t, t)$ denote the value of the derivative security at time t.

(b) Use risk-neutral valuation to calculate the price of this derivative security at time t.

(c) Confirm that your price satisfies the Black-Scholes differential equation.

$$\frac{\partial H(S_t, t)}{\partial t} + \frac{\partial H(S_t, t)}{\partial S_t}rS_t + \frac{1}{2}\frac{\partial H^2(S_t, t)}{\partial S_t^2}\sigma^2 S_t^2 - rH(S_t, t) = 0$$

Assume you are a trader and have sold the above security, and have the obligation to pay $(S_T)^3$ at time T. You have access to trade the underlying stock, call options and put options on the underlying stocks with various strikes, and lending/depositing money to the risk free rate.

Problem 2 Short term interest rate

We consider the the short term continuous interest rate r_t accrued on a risk free bank account β_t with initial value $\beta_0 = 1$.

(a) Explain why β_t follows the differential equation.

$$d\beta_t = r_t \beta_t dt$$

and integrate the equation to determine an integral form for β_t

(b) Determine the price $P(t,T)$ at time $t > 0$ of a zero coupon bond with maturity $T > t$.

(c) Give a general description of one-factor models of the interest rate, and give an example.

We assume there is only one factor driving the short rate, zero coupon prices are affine in the short rate and we use the traditional risk neutral world.

(d) Explain why the price $P(t,T)$ of a zero coupon bond with maturity T follows a stochastic differential equation of the form:

$$dP(t,T) = r_t P(t,T)dt + \sigma(r_t, t, T)P(t,T)dz_t \qquad P(T,T) = 1$$

where σ is the product of the volatility of the short rate and a function depending on time to maturity of the zero coupon bond.

Problem 3 — Credit derivatives

(a) Explain how to price a corporate bond in terms of hazard rate, probability of default, recovery rate and risk free rate.

Assume we have two companies, who both have issued corporate bonds with maturity T.

Company A has a probability of defaulting before time T of 20% whereas company B has a probability of defaulting on 40%. If any of them defaults the recovery rate is expected to be 40%, the amount will be paid to the investors at time T.

The risk free zero coupon bond paying one unit at time T has a price of 0.75.

(b) Explain what it means, that all above measures are under the risk free probability measure $\mathbb{Q}$

(c) Determine the price of the credit bonds of the two companies.

(d) Describe the characteristics of a CDS (Credit default swap) and find the prices the CDS's on the two companies with maturity T.

Now we introduce a first to default and second to default swap on the basket of company A and company B. A n-th to default swap means, if you have bought protection, you will have the same pay out as if you had bought protection on the n-th company in the basket defaulting.

(e) Argue that the portfolio of a first to default swap and a second to default swap on the basket of company 1 and company 2 must have the same price as the CDS's of the two companies.

Let the n-th to default swap have a price of Probability of pay out * (1-Recovery Rate) * Discount factor.

(f) Find the minimum price and the maximum price for the second to default swap, and the corresponding price of the first to default swap.

Now there probability of default before time T increases to 40% for company A, and to 60% for company B.

(g) Determine the new minimum and maximum prices of the first and second to default swap with the new default probabilities.

(h) Use the above example to give a short description of what happened in the structured credit markets during the recent financial crises for investors buying AAA rated structured notes, i.e. selling protection on a lot of companies defaulting at the same time.